Centro di Ricerca Matematica Ennio De Giorgi

Surfaces of General Type

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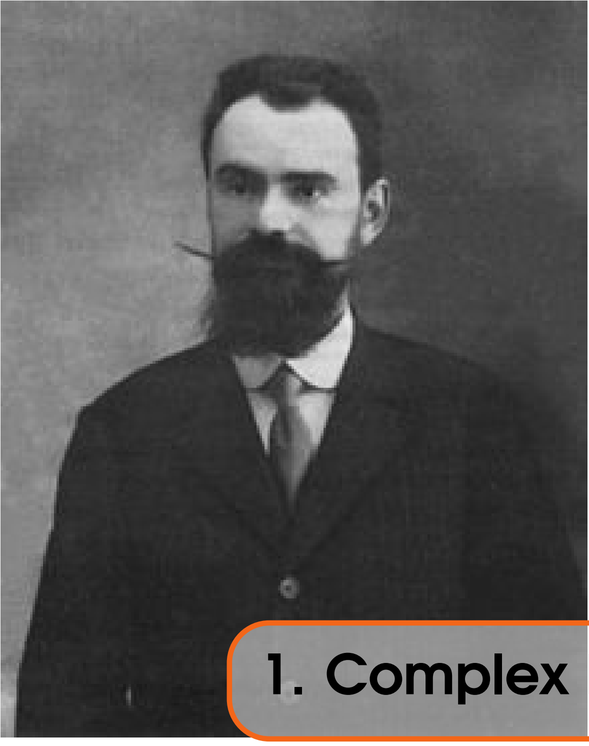
Contents



1. Complex Projective Surfaces

1.1 Notation and preliminaries

In this section we fix some notations and some basic results (we do not prove: good references are [Bea78] and [BPV84]) we will use in these lectures.

Definition 1.1.1 A **surface** (resp. **curve**) is a complex projective surface (resp. curve), that is an irreducible and reduced algebraic variety of dimension 2 (resp. 1) over the field of the complex numbers. We will mostly deal with **smooth** surfaces.

Definition 1.1.2 A **curve** C **in a smooth surface** S is a subscheme of codimension 1, so locally defined by one equation. In other words, curves in smooth surfaces are effective Cartier divisors. So a curve in a surface can be both reducible and not reduced.

To each curve (or more generally to each Cartier divisor) C corresponds a line bundle $\mathscr{O}_S(C)$ on S, and therefore a class in $H^1(\mathscr{O}_S^*)$; we will usually identify C with the image of that class by the map $c_1 \colon H^1(S, \mathscr{O}_S^*) \to H^2(S, \mathbb{Z})$ in the long cohomology exact sequence associated to the short exact sequence $0 \to \mathbb{Z} \to \mathscr{O}_S \to \mathscr{O}_S^*$.

Definition 1.1.3 The cup product on a smooth projective surface S give a symmetric bilinear form $H^2(S,\mathbb{Z}) \times H^2(S,\mathbb{Z}) \to \mathbb{Z}$.

The submodule $\operatorname{Im} c_1 \subset H^2(S,\mathbb{Z})$ is the **Neron-Severi** group of S, and denoted by $\mathrm{NS}(S)$. The **intersection product** of two curves (or more generally two effective divisors) C and D is the cup product of their classes in $\mathrm{NS}(S)$. We will denote it by CD or $C \cdot D$.

Definition 1.1.4 Let S be a smooth surface, and let A and B be two divisors on it. Then A and B are **numerically equivalent** if their classes in $\mathrm{NS}(S) \otimes_{\mathbb{Z}} \mathbb{R}$ are equal (equivalently: if $AC = BC$ for every curve C in S).

To compute it in most cases one needs only to know that

- if C and C' are linearly equivalent divisors, then they define the same class in $H^2(X,\mathbb{Z})$ and therefore they are also numerically equivalent;
- If $f \colon S \to B$ is a morphism of a surface onto a smooth curve, $\forall p \in B$ we define by F_p the

fibre f^*p. Then $\forall p, p'$, $c_1(F_p) = c_1(F_{p'})$ and therefore F_p and F_p' are numerically equivalent. In this case we will usually write F for the class of each F_p in $H^2(X,\mathbb{Z})$: note $F^2 = 0$;

- if C and D are irreducible distinct curves, they intersect in finitely many points and $CD = \sum_{p\in C\cap D}\mu(p,C,D)$, where $\mu(p,C,D)\in\mathbb{N}$, $\mu(p,C,D)\geq 1$ and $\mu(p,C,D) = 1$ if and only if C and D are smooth in p and transversal; in particolar if C and D are curves with no common components, then $CD \geq 0$ and $CD = 0$ if and only if $C\cap D = \emptyset$;
- if C is an ample divisor, then $CD > 0$ for every curve D.

and then argue by linearity.

A key tool in the study of projective surfaces is the following

Theorem 1.1.1 — Hodge Index Theorem. Let S be a smooth surface and consider $V := NS(S)\otimes_{\mathbb{Z}}\mathbb{R}$ endowed with the quadratic form induced by the intersection pairing. Define the **Picard number** of S as $\rho(S) := \dim_{\mathbb{R}} V$. Then the signature of this quadratic form in $(1,\rho-1)$.

Recall that on smooth varieties there are divisors K_X (the **canonical divisors**) such that $\omega_X := \mathscr{O}_X(K_X)$ is a dualizing sheaf for X.

Theorem 1.1.2 — Adjunction formula. If X is a Cohen-Macaulay variety and D is an effective Cartier divisor on X then $\omega_D = \omega_X(D)\otimes\mathscr{O}_D$ is a dualizing sheaf for D.

We will need the following classical result for surfaces

Theorem 1.1.3 — Riemann-Roch for surfaces. If S is a smooth surface and D is a divisor on S, then

$$\chi(\mathscr{O}_S(D)) = \chi(\mathscr{O}_S) + \frac{D(D-K_S)}{2}$$

which implies the genus formula.

Definition 1.1.5 If C is a curve on a surface S, we denote by $p_a(C)$ the **arithmetic genus** $p_a(C) = 1-\chi(\mathscr{O}_C)$

Note that if C is smooth irreducible, then this is exactly the genus of C.

Corollary 1.1.4 — Genus formula. If C is a curve on a smooth surface then $K_SC + C^2 = 2p_a(C)-2$

Proof. By the exact sequence $0\to\mathscr{O}_S(-C)\to\mathscr{O}_S\to\mathscr{O}_C\to 0$ follows $\chi(\mathscr{O}_C) = \chi(\mathscr{O}_S)-\chi(\mathscr{O}_S(-C))$. ■

1.2 Minimal surfaces

1.2.1 The blow-up

Consider $\mathbb{C}^{n+1}\times\mathbb{P}^n$ with the affine coordinates $(t_0,t_1,\dots,t_n)$ on the first factor and projective coordinates $(x_0,x_1,\dots,x_n)$ on the second factor. Then

$$(\mathbb{C}^{n+1})' = \{t_ix_j = t_jx_i\}$$

is a smooth complex manifold containing the divisor $E = \{(0,\dots,0\}\times\mathbb{P}^n\cong\mathbb{P}^n$, and the projection on the first factor give a birational morphism. $\pi_1\colon(\mathbb{C}^{n+1})'\to\mathbb{C}^{n+1}$, contracting E to the origin, and mapping biregularly $(\mathbb{C}^{n+1})'\setminus E$ onto $\mathbb{C}^{n+1}\setminus\{(0,\dots,0)\}$.

Then $(\mathbb{C}^{n+1})'$ and the pair $((\mathbb{C}^{n+1})',\pi_1)$ are the **blow-up** of $\mathbb{C}^{n+1}$ at $\{0\}$.

By glueing charts, one immediately generalizes this procedure to the blow-up of smooth algebraic variety (or complex manifold) X at any point p, getting a new smooth algebraic variety

X', the **blow-up** of X at p, containing a smooth effective divisor $E \cong \mathbb{P}^{\dim X-1}$, the **exceptional divisor** and a morphism $\pi \colon X' \to X$ contracting E to p and mapping biregularly $X' \setminus E$ to $X \setminus p$.

Theorem 1.2.1 If X is projective, then X' is projective too.

If moreover $\dim X = 2$, then
- $\forall m \geq 0$, $|mK_{X'}| = \pi^*|mK_X| + mE$;
- every divisor in S' is linear equivalent to a divisor of the form $\pi^*C + \lambda E$, $\lambda \in \mathbb{Z}$ so that we can write
$$\mathrm{NS}(X') \cong \mathrm{NS}(X) \oplus^{\perp} \mathbb{Z}E;$$
- for every pair of divisors C and D on X $(\pi^*C)\cdot(\pi^*D) = C \cdot D$, $E\pi^*C = 0$;
- $E^2 = K_{X'}E = -1$.

Definition 1.2.1 Let $\pi \colon Y \to X$ be the blow up in a point with exceptional divisor E, let D be a curve in X. Then π^*D can be written uniquely as $\pi^*D = \tilde{D} + dE$ for some $d \geq 0$ so that $\tilde{D}$ is effective and E is not a component of $\tilde{D}$. $\tilde{D}$ is the **strict transform** of D.

It can be shown (see Exercises 1.1 and 1.2) that
1) $p_a(\tilde{D}) \leq p_a(D)$;
2) $p_a(\tilde{D}) = p_a(D)$ if and only if $p \notin D$ or p is a smooth point of D: in both cases $\pi_{|\tilde{D}} \colon \tilde{D} \to D$ is an isomorphism;
3) if D is reduced, then after finitely many suitable blow-ups its strict transform is smooth.

These results togheter give

Corollary 1.2.2 Let C be an irreducible curve in a smooth surface S. Then $p_a(C) \geq 0$ (equivalently $K_SC + C^2 \geq -2$). If moreover $K_SC + C^2 = -2$, then C is smooth and rational (that is $C \cong \mathbb{P}^1$).

Blow-up's are often use to transform rational maps in morphisms as follows.

Theorem 1.2.3 — Resolution of rational maps. Let S be a smooth surface, and consider a rational map $f \colon S \dashrightarrow \mathbb{P}^n$. Then there is a finite sequence of blow-ups $\varepsilon \colon S^{(r)} \to S^{(r-1)} \to \cdots \to S' \to S$ and a morphism $g \colon S^{(r)} \to \mathbb{P}^n$ such that the diagram

$$\begin{tikzcd} & S^{(r)} \arrow[dl, "\varepsilon"'] \arrow[dr, "g"] & \\ S \arrow[rr, dashed, "f"'] & & \mathbb{P}^n \end{tikzcd} \tag{1.1}$$

commutes.

g is a **resolution** of the indeterminacy locus of f. The resolution is **minimal** if r is the minimum possible number among all possible resolutions of the indeterminacy locus of f.

It is easy to detect if a surface is a blow-up of an other surfaces.

Theorem 1.2.4 — Castelnuovo contractibility theorem. Let S' be a smooth surface and E a smooth rational curve on S' such that $E^2 = -1$. Then there exist a smooth surface S and a morphism $\pi \colon S' \to S$ such that π contracts E to a point p and (S', π) is isomorphic to the blow-up of S at p.

This motivates the definition of **minimal** surface, which is a surface that is not isomorphic to the blow-up of any other surface.

Definition 1.2.2 A smooth surface is minimal if it does not contain any smooth rational curve E with $E^2 = -1$.

An immediate consequence of this definition is the

Proposition 1.2.5 Every smooth surface S is birational to a minimal surface.

Proof. If S is not minimal, it has a smooth rational curve E with $E^2 = -1$, and contracting it we get a surface S_1 with $\operatorname{rank} \mathrm{NS}(S_1) = \operatorname{rank} \mathrm{NS}(S) - 1$. If S_1 is not minimal, we repeat the procedure constructing a new surface S_2 and so on. Since $\operatorname{rank} \mathrm{NS}(S) < \infty$, the procedure terminates. ■

1.3 Enriques classification

From the point of view of classification theory, since we know that every surface is obtained by a minimal one by finitely many blow-ups, and the blow-up is a rather simple procedure, it is natural then to restrict itself to the study of minimal surfaces. A key role in this study is played by the following numbers.

Definition 1.3.1 Let S be a smooth surface. We associate to S the following numbers, who are birational invariants.

- the **geometric genus** $p_g(S) := h^0(\mathscr{O}_S(K_S))$
- the **m-th plurigenus** $P_m := h^0(\mathscr{O}_S(mK_S))$
- the **irregularity** $q := h^1(\mathscr{O}_s) = h^0(\Omega^1_S)$ (last equality follows by Hodge theory)
- the **Euler characteristic** $\chi := \chi(\mathscr{O}_S) = 1 - q + p_g$

The reason why most of the numbers above are birational invariants, is by the fact that, if $\pi \colon Y \to X$ is a blow-up, $|mK_Y| = \pi^*|mK_X| + mE$.

Definition 1.3.2 Let S be a smooth surface. Its **canonical ring** is the graded ring

$$R := \oplus_{d \geq 0} H^0(\mathscr{O}_S(dK_S))$$

with product given by the tensor product of sections (here the homogeneous piece R_d of degree d is clearly $H^0(\mathscr{O}_S(dK_S))$).

Then by the argument above birational surfaces have isomorphic canonical rings. The plurigenera give the Hilbert function of R. The growth of them define then a further birational invariant

Definition 1.3.3 Let S be a be a smooth surface. Its Kodaira dimension is

$$\kappa(S) = \min\left(k \middle| \left\{\frac{P_d(s)}{d^k}\right\} \text{ is bounded from above}\right)$$

When all plurigenera vanish, one conventially set $\kappa(S) = -\infty$.

Theorem 1.3.1 — Uniqueness of the minimal model. Let S and S' be two minimal surfaces, and assume that there is a birational map $f \colon S \dashrightarrow S'$. Assume $\kappa(S) \neq -\infty$. Then f is biregular.

Recall that a divisor D on S is **nef** if for every irreducible curve C in S, $DC \geq 0$.

Theorem 1.3.2 Let S be a surface. If K_S is nef then S is minimal.
If $\kappa(S) \neq -\infty$, then S is minimal if and only if K_S is nef.

Proof. If S is not minimal, then there is a rational curve E in S with $K_S E = -1$, so K_S is not nef.
Assume then $\kappa(S) \neq -\infty$, so there is an effective divisor $D \in |mK_S|$ for some $m > 0$.

If K_S is not nef, then there is an irreducible curve C in S with $DC < 0$. Writing $D = \sum d_i D_i$ we see that $\exists i$ with $CD_i < 0$, so $C = D_i$ and $C^2 < 0$. Now $C^2 \leq -1$, $K_S C \leq -1$ so $C^2 + K_S C \leq -2$. Since C is irreducible, by the genus formula $p_a(C) = 0$, so C is smooth rational and $C^2 = KC = -1$. Then S is not minimal. ■

There is the following classification

Theorem 1.3.3 — Enriques[a] classification. Let S be a smooth minimal surface. Then S is one of the following.

- $\kappa = -\infty$: $\mathbb{P}^2$;
- $\kappa = -\infty$: a **ruled**[b] surface: a surface S fibred as $S \to B$ onto a smooth curve B such that all fibres are isomorphic to $\mathbb{P}^1$;
- $\kappa = 0$: a **K3** surface: a simply connected surface with $\mathcal{O}_S(K_S) \cong \mathcal{O}_S$, $q = 0$;
- $\kappa = 0$: an **Enriques**[c] surface: a surface with $\mathcal{O}_S(K_S) \ncong \mathcal{O}_S$, $\mathcal{O}_S(2K_S) \cong \mathcal{O}_S$, $q = 0$;
- $\kappa = 0$: an **abelian** surface: a quotient $\mathbb{C}^2_{/\Lambda}$ by a lattice Λ of rank 4: $\mathcal{O}_S(K_S) \cong \mathcal{O}_S$, $q = 2$;
- $\kappa = 1$: a[d] minimal **elliptic** surfaces: a surface fibred as $S \to B$ onto a smooth curve B such that the general fibre is smooth of genus 1 (these have $K^2 = 0$);
- $\kappa = 2$: a minimal surface **of general type**.

[a]This classification has been extended in the ′60s by Kodaira to all compact complex manifold of dimension 2, including the non-algebraic compact surfaces. That generalization is known as Enriques-Kodaira classification.

[b]There is exactly one ruled surface, the Hirzebruch surface $\mathbb{F}_1$, which is not minimal; all other ruled surfaces are minimal surfaces with $\kappa(S) = -\infty$

[c]These have $\pi_1(S) = \mathbb{Z}_{/2\mathbb{Z}}$: their universal cover is a K3 surface.

[d]not all elliptic surfaces have $\kappa(S) = 1$; they may have also $\kappa(S) = 0$ or $\kappa(S) = -\infty$. For example, all Enriques surfaces are elliptic.

The last line of Theorem 1.3.3 is just a definition:

Definition 1.3.4 A surface S is of general type if $\kappa(S) = 2$.

■ **Example 1.1 — Product of two curves.** Let C_1, C_2 be two curves of genus $g(C_i) =: g_i \geq 2$. Then $C_1 \times C_2$ is minimal of general type with $q = g_1 + g_2$, $p_g = g_1 g_2$, $K^2 = 4(g_1 - 1)(g_2 - 1)$. ■

■ **Example 1.2 — Hypersurfaces in a projective space.** Fix $d \geq 5$. Let S be a smooth divisor in $|\mathcal{O}_{\mathbb{P}^3}(d)|$. Then S has $q = 0$, $\omega_S = \mathcal{O}_S(d-4)$, $K_S^2 = d(d-4)^2$, $p_g = \binom{d-1}{3}$.

Then ω_S is nef and so S is minimal. Then $K_S^2 > 0$ implies $\forall m \geq 2$, $h^2(mK_S) = h^0((1-m)K_S) = 0$, and then by Riemann-Roch $P_m(S) \geq \chi(\mathcal{O}_S(mK_S)) = \chi(\mathcal{O}_S) + \frac{m(m-1)}{2} K_S^2$, so $\kappa(S) = 2$.

Similarly complete intersections of $n-2$ hypersurfaces in $\mathbb{P}^n$ are *almost always* minimal of general type. ■

■ **Example 1.3 — Godeaux[1] surfaces.** Consider the Fermat quintic $\{x_1^5 + x_2^5 + x_3^5 + x_4^5 = 0\} \subset \mathbb{P}^3$, it is a smooth minimal surface of general type with $\omega_S = \mathcal{O}_S(1)$, $q = 0$, $p_g = 4$, $K_S^2 = 5$.

Set $\eta := e^{\frac{2\pi i}{5}}$ and let $\mathbb{Z}/5\mathbb{Z}$ act on $\mathbb{P}^3$ by $(x_1, x_2, x_3, x_4) \mapsto (\eta x_1, \eta^2 x_2, \eta^3 x_3, \eta^4 x_4)$. Note that $\mathbb{Z}/5\mathbb{Z}$ acts on S, and the action on S is free, so that $S' := S_{/\mathbb{Z}/5\mathbb{Z}}$ is a smooth surface and the projection $\pi\colon S \to S'$ is étale of degree 5.

First note (for example by the Lefschetz fixed point formula, as the group has order 5 and acts freely) $\chi(\mathcal{O}_S) = 5\chi(\mathcal{O}_{S'})$. So $\chi(\mathcal{O}_{S'}) = \frac{5}{5} = 1$.

Moreover $\Omega^1(S) = \pi^*\Omega^1(S')$, and then (since we know $q(S) = 0$) $q(S') = 0$. So $p_g(S) = 0$.

Similarly $K_S = \pi^* K_{S'}$: note that this implies that $K_{S'}$ is nef, and $K_{S'}^2 = \frac{5}{5} = 1 > 0$. So S is of general type. ■

[1]This construction, given by Godeaux in the 30s, is one of the first examples of surfaces of general type with $p_g = 0$.

Exercise 1.1 Show that, if $\tilde{D}$ is the strict transform of a curve D in a surface by the blow-up in a point, then $p_a(\tilde{D}) \leq p_a(D)$ ■

Exercise 1.2 Show[a] that, if $\tilde{D}$ is the strict transform of D in a surface by the blow-up in a point p, then $p_a(\tilde{D}) = p_a(D)$ if and only if $p \notin D$ or p is a smooth point of D. ■

[a]Hint: writing $\pi^*D = D + mE$ show that p is a smooth point of D if and only if $m = 1$

Exercise 1.3 — Enriques surfaces. Consider a smooth complete intersection of three quadrics $S = Q_0 \cap Q_1 \cap Q_2 \subset \mathbb{P}^5$. Show that it is a minimal surface, and more generally a $K3$ surface.

Let $\mathbb{Z}/2\mathbb{Z}$ act on $\mathbb{P}^5$ by $(x_0,x_1,x_2,x_3,x_4,x_5) \mapsto (x_0,x_1,x_2,-x_3,-x_4,-x_5)$. Assume that all Q_i are of the form $\sum a_{ij}x_j^2 = 0$; then $\mathbb{Z}/2\mathbb{Z}$ acts on S.

Show that if Q_0, Q_1 and Q_2 are general, then the action on S is free, and $S' := S_{/\mathbb{Z}/2\mathbb{Z}}$ is an Enriques surface. ■

Exercise 1.4 — Campedelli[a] surfaces. Consider a smooth complete intersection of four quadrics $S = Q_0 \cap Q_1 \cap Q_2 \cap Q_3 \subset \mathbb{P}^6$. Show that it is a minimal surface of general type with $q = 0$, $p_g = 7$, $K_S^2 = 16$.

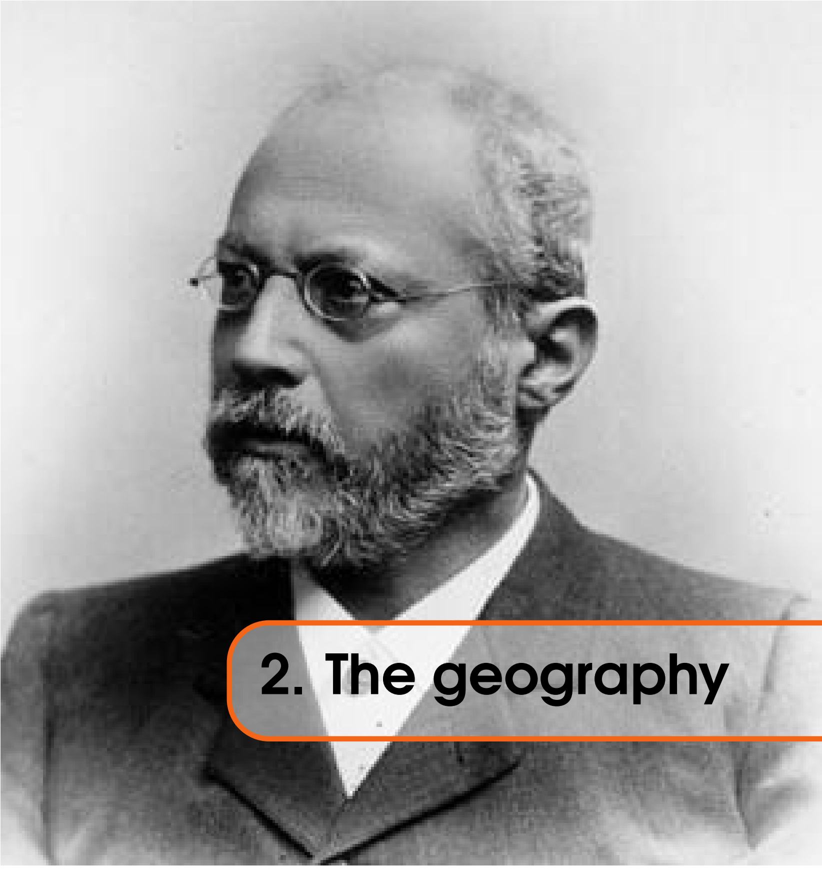
Let $(\mathbb{Z}/2\mathbb{Z})^3$ act on $\mathbb{P}^6$ by

$$(a,b,c)(x_0,x_1,x_2,x_3,x_4,x_5,x_6) = \\ = ((-1)^a x_0, (-1)^b x_1, (-1)^c x_2, (-1)^{a+b} x_3, (-1)^{a+c} x_4, (-1)^{b+c} x_5, (-1)^{a+b+c} x_6).$$

Assume that all Q_i are of the form $\sum a_{ij}x_j^2 = 0$; then $(\mathbb{Z}/2\mathbb{Z})^3$ acts on S.

Show that if Q_0, Q_1, Q_2 and Q_3 are general, then the action on S is free, and $S' := S_{/(\mathbb{Z}/2\mathbb{Z})^3}$ is a minimal surface of general type with $K_S^2 = 2$, $p_g = q = 0$. ■

[a]These surfaces have been constructed by Campedelli in the 30s, more or less at the same time of Godeaux construction, but this construction is not Campedelli's one



2. The geography

2.1 Improving "K is nef" on minimal surfaces of general type

If S is a minimal surface of general type, then by Theorem 1.3.2, K_S is nef. Since by definition $|nK_S|$ is not empty for large n, follows immediately $K_S^2 \geq 0$. A slightly better inequality holds.

Proposition 2.1.1 Let S be a minimal surface of general type. Then $K_S^2 \geq 1$.

Proof. Let H be a general (then smooth) hyperplane section of S. As nK_S is effective for large n, $HK_S > 0$. Consider the exact sequence

$$0 \to \mathcal{O}_S(nK_S - H) \to \mathcal{O}_S(nK_S) \to \mathcal{O}_H(nK_S) \to 0$$

for large n, and its long cohomology exact sequence. By the Riemann-Roch theorem for curves $h^0(\mathcal{O}_H(nK_S))$ grows linearly with n whereas by assumption P_n grows more quickly. So for large n there is an effective divisor in $|nK_S - H|$, and then $(nK_S - H)K_S \geq 0$, so $nK_S^2 \geq HK_S > 0$. ■

Corollary 2.1.2 Let S be a minimal surface of general type, then $h^1(\mathcal{O}_S(nK_S)) = 0$ for all $n \neq \{0, 1\}$.

Proof. The case $n < 0$ follows by Mumford's vanishing theorem (if D is nef and $D^2 > 0$ then $h^1(\mathcal{O}_S(-D)) = 0$). The case $n \geq 2$ follows then by Serre duality. ■

We can improve the assertion that K_S is nef in a different direction.

Proposition 2.1.3 Let S be a minimal surface of general type. Then[1] the irreducible curves C in S with $K_S C = 0$ are all smooth and rational, and they are at most $\rho(S) - 1$.

Moreover the symmetric matrix $(C_i \cdot C_j)$ is negative definite and then their classes form a linearly independent set $\{C_1, \ldots, C_k\}$ in $\mathrm{NS}(S) \otimes_{\mathbb{Z}} \mathbb{R}$.

[1] This proof comes from [Bom73].

Proof. Let C be an irreducible curve with $K_S C = 0$, so its class in $\mathrm{NS}(S) \otimes_{\mathbb{Z}} \mathbb{R}$ belongs to $\langle K_S \rangle^{\perp}$. By Proposition 2.1.1 and the Hodge Index Theorem 1.1.1 follows then $C^2 \leq 0$ and $C^2 = 0$ if and only if C is numerically trivial which is impossible as C is effective (so $CH > 0$ for any hyperplane section H). So $C^2 < 0$. Then by the genus formula $p_a(C) = 1 + \frac{1}{2}(C^2 + K_S C) = 1 + \frac{1}{2}C^2 < 1$, so $p_a(C) = 0$ and C is smooth and rational with $C^2 = -2$.

Now assume that $C_1, \ldots, C_r$ are distinct irreducible curves with $K_S C_i = 0$, not linearly independent in $\mathrm{NS}(S) \otimes_{\mathbb{Z}} \mathbb{R}$. Then we can find constants $c_i > 0$ so that, for some $1 < k < r$, $A = \sum_{i \leq k} c_i C_i$ and $B = \sum_{i \geq k+1} c_i C_i$ are numerically equivalent. But then $A^2 = AB \geq 0$ contradicts (arguing as above) the Hodge Index Theorem 1.1.1, since $A \in \langle K_S \rangle^{\perp}$ is effective. ■

2.2 Noether's inequality

Definition 2.2.1 A projective variety $X \subset \mathbb{P}^n$ is **nondegenerate** if it is not contained in any linear subspace.

R The image of a variety by the rational map induced by a linear system is always nondegerate.

We need a classical result on the degree of a nondegenerate projective surface.

Lemma 2.2.1 Let $\Sigma \subset \mathbb{P}^n$ be a nondegenerate surface, and let d be its degree. Then $d \geq n-1$.

If moreover Σ is not ruled, then $d \geq 2(n-1)$, and K_Σ is numerically trivial[2] if equality holds.

Theorem 2.2.2 — Noether inequality[a]. Let S be a minimal surface of general type. Then $K_S^2 \geq 2p_g(S) - 4$. If the equality holds, then $\varphi_{|K_S|}$ is a degree 2 morphism onto a nondegenerate surface of minimal degree $p_g - 2$ in $\mathbb{P}^{p_g - 1}$.

[a] Some people denote as Noether inequality the slightly weaker inequality $K_S^2 \geq 2\chi(\mathcal{O}_S) - 6$. The proof here is essentially taken by [Sak80].

Proof. By $K_S^2 \geq 1$ we can assume $p_g(S) \geq 3$.

Let Z be the fixed part of $|K_S|$, so we can write $|K_S| = |D| + Z$ where D has no fixed components.

Since $p_g(S) \geq 3$ we may consider the canonical map $\varphi_{|K_S|} \colon S \dashrightarrow \mathbb{P}^{p_g - 1}$. Let $\pi \colon S^* \to S$ be the blow up of the indeterminacy locus of $|D|$ so that the movable part $|L|$ of $|\pi^* D|$ (which is also the movable part of $|K_{S^*}|$) is base point free. We get then a morphism

$$\varphi_{|K_S|} \circ \pi = \varphi_{|L|} \colon S^* \to \Sigma$$

Let Σ be its image $\varphi_{|K_S|}(S)$: it is an irreducible subvariety of $\mathbb{P}^{p_g - 1}$, $p_g \geq 3$, which is nondegenerate. So $\dim \Sigma \in \{1, 2\}$.

We first consider the case $\dim \Sigma = 1$. The Stein factorization of $\varphi_{|L|}$ is

$$S^* \xrightarrow{p} B \xrightarrow{\theta} \Sigma$$

where B is a smooth curve, p has connected fibres and θ is a finite map.

Let H be an hyperplane section of Σ, and let n be the degree of $\theta^* H$. Then

$$p_g(S) = p_g(S^*) = h^0(\mathcal{O}_{S^*}(L)) = h^0(\mathcal{O}_{S^*}(p^*\theta^* H)) = h^0(\mathcal{O}_B(\theta^* H))$$

[2] Here Σ is not necessarily smooth, but under these assumptions one can show that there is a Cartier divisor K_Σ such that $\mathcal{O}_\Sigma(K_\Sigma)$ is a dualizing sheaf for Σ and moreover the class of Σ in $\mathrm{NS}(\Sigma) \otimes_{\mathbb{Z}} \mathbb{R}$ is zero.

and then, denoting by g the genus of B, by Riemann-Roch theorem

$$p_g(S) = n+1-g \text{ if } n > 2g-2$$

and, by Clifford theorem

$$p_g(S) \leq \frac{1}{2}n+1 \text{ if } n \leq 2g-2.$$

The two claims together give

$$p_g(S) \leq n+1. \tag{2.1}$$

On the other hand, denoting by F^* a general fibre of p, and by F its image on S, D is numerically equivalent to nF, and then

$$K_S^2 = K_S(nF+Z) \geq nK_SF = n(nF^2+ZF)$$

where the inequality follows from K_S nef.

We claim $nF^2+ZF \geq 2$, which immediately implies

$$K_S^2 \geq 2n. \tag{2.2}$$

We prove the claim. Since D has no fixed components, $D^2 \geq 0$, $DZ \geq 0$, and therefore $F^2 \geq 0$, $FZ \geq 0$. Since Σ is nondegenerate, $n \geq \deg\Sigma \geq 2$ and then our claim follows if we exclude the case $F^2 = 0$, $FZ \in \{0,1\}$ Indeed, if $F^2 = 0$, by the genus formula $ZF = K_SF$ is even, thus excluding $ZF = 1$. Finally, if $ZF = F^2 = 0$, then $F \in \langle K_S\rangle^{\perp}$, contradicting Proposition 2.1.3.

Finally (2.1) and (2.2) together give the inequality $K_S^2 \geq 2p_g - 2$, slightly[3] strictly stronger than the stated inequality, concluding the case $\dim\Sigma = 1$ (equality can't occur).

We can then assume $\dim\Sigma = 2$. Arguing as above,

$$K_S^2 = D^2+DZ+K_SZ \geq D^2 \geq L^2 = (\deg\varphi_{|K_S|})(\deg\Sigma) \tag{2.3}$$

where the last equality comes from $L = \varphi^*_{|K_S|}(H)$ for a hyperplane section H of Σ. We have two cases.

1) If $\deg\varphi_{|K_S|} = 1$, then Σ is birational to a surface of general type, and then neither it can be ruled[4] nor K_Σ can[5] be numerically trivial. By Lemma 2.2.1, $\deg\Sigma > 2(p_g-1)-2$. Then by (2.3) $K_S^2 > 2p_g-4$, stronger than required. In this case the equality cannot occur.
2) Else $\deg\varphi_{|K_S|} \geq 2$ and then (2.3) and Lemma 2.2.1 give $K_S^2 \geq 2p_g-4$. Here the equality may occur when $\deg\varphi_{|K_S|} = 2$ and Σ has minimal degree. Moreover, if equality occurs it must occur also in all inequalities of (2.3): in particular from $D^2 = L^2$ it follows that $\varphi_{|K_S|}$ is a morphism (in other words $S^* = S$).

■

2.3 The geography

There are two more inequalities among the invariants of a surface of general type.

[3]When $\dim\Sigma = 1$ a much stronger inequality has been proved by Xiao Gang in [Xia85]: indeed in this case $K_X^2 \geq 4p_g-6$ unless X is one of the surfaces with $p_g = 2$ and $K^2 = 1$ in the Example 2.2. The Xiao inequality is sharp, as the equality can be realized for every value of p_g; the surfaces with $K^2 \leq 4p_g-4$ and $\dim\Sigma = 1$ have been classified in [Pig12].

[4]If Σ is ruled, then S is covered by rational curves, which implies that $\kappa(S) = -\infty$, compare Theorem 1.3.3.

[5]One can show $K_{S^*} \leq \varphi^*_{|L|}K_\Sigma$

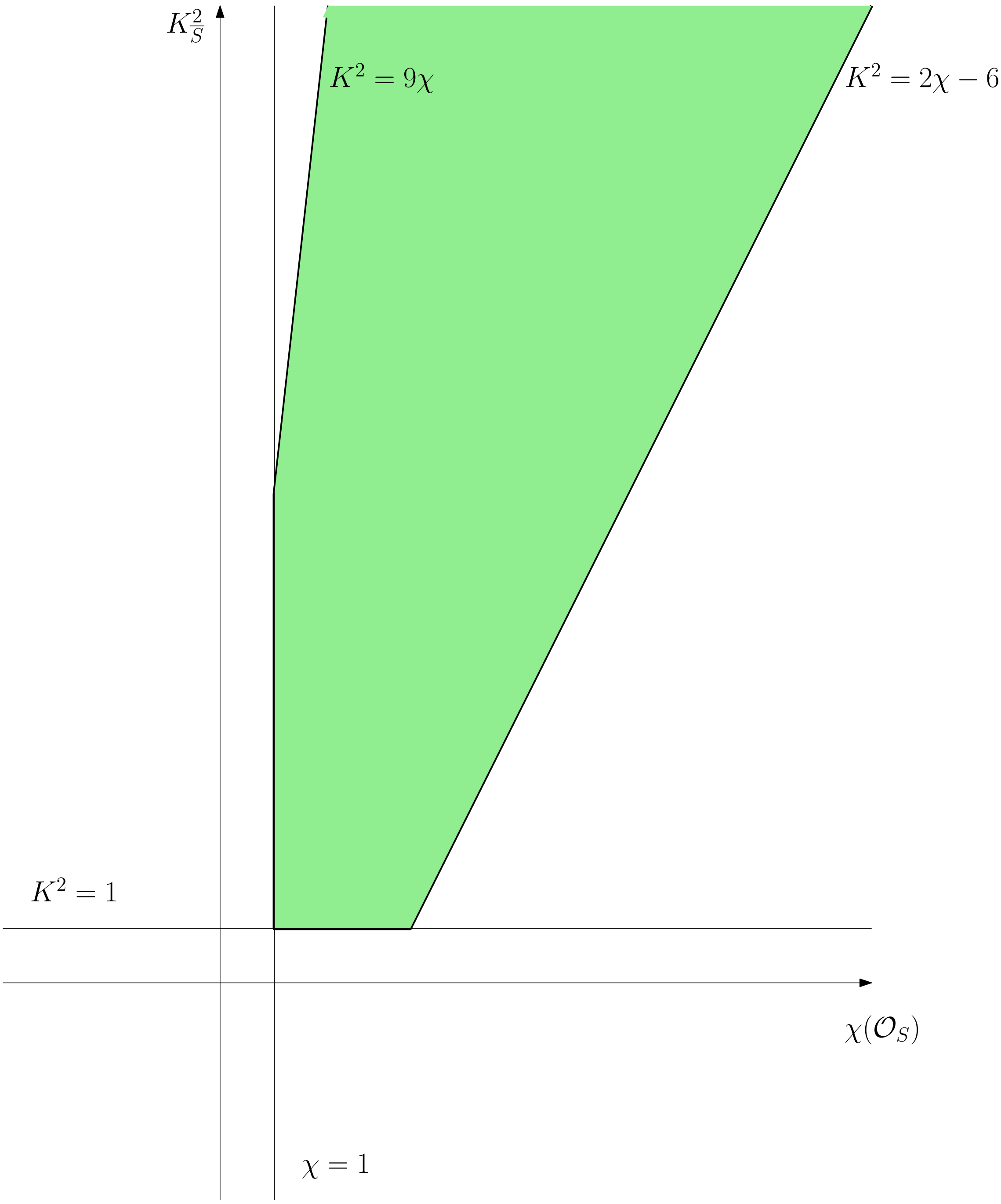


Figure 2.1: The *geography* of the surfaces of general type

Theorem 2.3.1 Let S be a surface of general type, then $\chi(\mathscr{O}_S) \geq 1$ and $K_S^2 \leq 9\chi$.

which, with Proposition 2.1.1 and Theorem 2.2.2, determines a quadrilateral region of the plane where the pair (K^2, χ) can stay: this is the region in Figure 2.1.

2.4 Weighted projective spaces: some surfaces on the Noether line

Let $(a_0, \dots, a_n) \in \mathbb{N}^{n+1}$. The **weighted projective space** $\mathbb{P} := \mathbb{P}(a_0, \dots, a_n)$ is defined as $\mathbb{P} := \mathrm{Proj}(A)$ where A is the polynomial ring $\mathbb{C}[x_0, \dots, x_n]$ graded so that $\deg x_i = a_i$. We will denote by A_d the vector subspace of the weighted homogeneous polynomials of degree d. The a_i are the **weights** of $\mathbb{P}$. We restrict to the *well-formed* case, *i.e.* assuming that each subset of n of the $n+1$ weights have no common divisors: for example the straight projective space $\mathbb{P}(1,1,1,1) \cong \mathbb{P}^3$ or $\mathbb{P}(1,1,2,5)$ (whereas $\mathbb{P}(1,2,2,2)$ is not well-formed, and we do not allow that).

They can be also seen as quotients $\left(\mathbb{C}^{n+1} \setminus \{0\}\right)/\mathbb{C}^*$ and precisely the quotient by the $\mathbb{C}^*$-action

$$\lambda(x_0, x_1, \dots, x_n) = (\lambda^{a_0} x_0, \lambda^{a_1} x_1, \dots, \lambda^{a_n} x_n).$$

The following are well known results on weighted projective spaces whose proofs are in [Dol82].

They are (usually singular) varieties, on which there are sheaves $\mathscr{O}_{\mathbb{P}}(d)$ defined analogously to the case of the *straight* projective spaces, although they are in general not locally free at the singular points of X: more precisely they are line bundles if and only if d is a multiple of $\mathrm{lcm}(a_i)$. Moreover

- $|\mathscr{O}_{\mathbb{P}}(\mathrm{lcm}(a_i))|$ is very ample;
- $\forall d \in \mathbb{N}$, $H^0(\mathscr{O}_{\mathbb{P}}(d)) \cong A_d$;
- for each $0 < i < n$, $\forall d$, $h^i(\mathscr{O}_{\mathbb{P}}(d)) = 0$;
- The dualizing sheaf of $\mathbb{P}$ is $\mathscr{O}_{\mathbb{P}}(-\sum a_i)$.

A weighted homogeneous polynomial $f \in A_d$ has a zero locus $V(f) \subset \mathbb{P}$ which is a Weil divisor, we will write $V(f) \in |\mathscr{O}_{\mathbb{P}}(d)|$. Given r weighted homogeneous polymonials $f_1, \dots, f_r$ their zero locus $V(f_1, \dots, f_r)$ is a **quasi-smooth complete intersection** if $\{f_1 = \cdots = f_r = 0\} \subset \mathbb{C}^{n+1} \setminus \{0\}$ is a smooth complete intersection. If a quasi-smooth divisor does not intersect the singular locus of $\mathbb{P}$, it is smooth.

If $X = V(f_1, \dots, f_r) \in |\mathscr{O}_{\mathbb{P}}(d)|$ is a quasi-smooth complete intersection, then

- $H^0(\mathscr{O}_X(d)) \cong (A/(f_1, \cdots, f_r))_d$;
- for each $0 < i < n-r-1$, $\forall d$, $h^i(\mathscr{O}_X(d)) = 0$;
- $\mathscr{O}_X(\sum \deg f_i - \sum a_j)$ is a dualizing sheaf for X.

■ **Example 2.1** Consider $\mathbb{P} := \mathbb{P}(1,1,1,4)$, and a smooth $X_8 \in |\mathscr{O}_{\mathbb{P}(1,1,1,4)}(8)|$, so $X = V(f)$ for $f = x_3^2 + x_3 f_4(x_0,x_1,x_2) + f_8(x_0,x_1,x_2)$.

By the formulas above $\omega_{X_8} = \mathscr{O}_{X_8}(8-1-1-1-4=1)$, so $\omega_{X_8}^4$ is very ample, and therefore ω_X is nef.

Moreover $p_g(X) = \dim A_1 = 3$, and $\forall m \in \mathbb{Z}$ $h^1(\mathscr{O}_{X_8}(mK_{X_8})) = 0$, so $q = 0$ and $P_2(X_8) = \dim A_2 = 6$ which give by Riemann Roch $K_{X_8}^2 = P_2 - 1 + q - p_g = 2$. Note that, since $h^1(\mathscr{O}_{X_8}(mK_{X_8})) = 0$ and $K_{X_8}^2 > 0$, by Riemann-Roch P_m grows quadratically, so X_8 is minimal (as K_X is nef) of general type.

Note that $K_{X_8}^2 = 2p_g(X_8) - 4$: this surface realizes the equality in Noether's inequality so by Theorem 2.2.2 $\phi_{|K_S|}$ is a degree 2 morphism on $\mathbb{P}^2$. Indeed φ_{K_S} is by construction the map $S \to \mathbb{P}^2$ given by the projection $(x_0,x_1,x_2,x_3) \dashrightarrow (x_0,x_1,x_2)$, that has degree 2. ■

■ **Example 2.2** Consider $\mathbb{P} := \mathbb{P}(1,1,2,5)$, with coordinates (x_0,x_1,y,z) and a smooth surface $X_{10} \in |\mathscr{O}_{\mathbb{P}}(10)|$. We can then see it as $X_{10} = V(f)$ for

$$f = z^2 + ay^5 + +x_0 g_0(x_0,x_1,y,z) + x_1 g_1(x_0,x_1,y,z)$$

By the formulas above $\omega_{X_{10}} = \mathcal{O}_{X_{10}}(10-1-1-2-5=1)$ is ample and then nef (as in the previous example), $p_g(X) = \dim A_1 = 2$, and moreover $\forall m$, $h^1(\mathcal{O}_{X_{10}}(mK_{X_{10}})) = 0$, so $q = 0$ and $P_2(X_{10}) = \dim A_2 = 4$ which give $K^2_{X_8} = P_2 - 1 + q - p_g = 1$.

In this case the image of the canonical map is $\mathbb{P}^1$, so it has dimension 1. Note that the canonical map is the restriction of $(x_0, x_1, y, z) \dashrightarrow (x_0, x_1)$, so it is not defined at the unique point in $\{x_0 = x_1 = 0\} \cap X_{10}$. ■

Exercise 2.1 Show that the surfaces in the Example 2.1 exist by using first a Bertini argument to show that the general $X_8 \in |\mathcal{O}_{\mathbb{P}}(8)|$ is quasi-smooth, and then by using that the only singular point of $\mathbb{P}(1,1,1,4)$ is $(0,0,0,1)$. ■

Exercise 2.2 Use a similar argument to show that the surfaces in the Example 2.2 exist[a].

In the notation of the proof of Theorem 2.2.2, these surfaces have $\dim \Sigma = 1$. At a first glance, they seems to be a counterexample to that part of the proof, as they violates the inequality $K^2 \geq 2p_g - 2$. But indeed, this is not true as we were assuming $p_g \geq 3$, whereas these surfaces have $p_g = 2$.

Find where exactly the proof of $\dim \Sigma = 1 \Rightarrow K^2 \geq 2p_g - 2$ fails for $p_g = 2$. ■

[a] The singular points of $\mathbb{P}(1,1,2,5)$ are $(0,0,1,0)$ and $(0,0,0,1)$

Exercise 2.3 Set $\mathbb{P} = \mathbb{P}(1,1,1,1,3)$ and choose two general hypersurfaces $Q \in |\mathcal{O}_{\mathbb{P}}(2)|$ and $G \in |\mathcal{O}_{\mathbb{P}}(6)|$.

Show[a] that, if Q and G are general enough, then $X_{12} := Q \cap G$ is a smooth minimal surface of general type. compute its invariants p_g. q and K^2_S and locate it in the geography. Describe its canonical map. ■

[a] In case you don't know, the singular locus of $\mathbb{P}$ is just the point $(0,0,0,0,1)$

Exercise 2.4 Set $\mathbb{P} = \mathbb{P}(1,1,1,2,2)$ and choose two general hypersurfaces $G_1, G_2 \in |\mathcal{O}_{\mathbb{P}}(4)|$.

1) Show[a] that, if G_1 and G_2 are general enough, then $X_{16} := G_1 \cap G_2$ is a smooth minimal surface of general type. Compute its invariants p_g. q and K^2_S and locate it in the geography.
2) Consider the action of $\mathbb{Z}/4\mathbb{Z}$ on $\mathbb{P}$ generated by

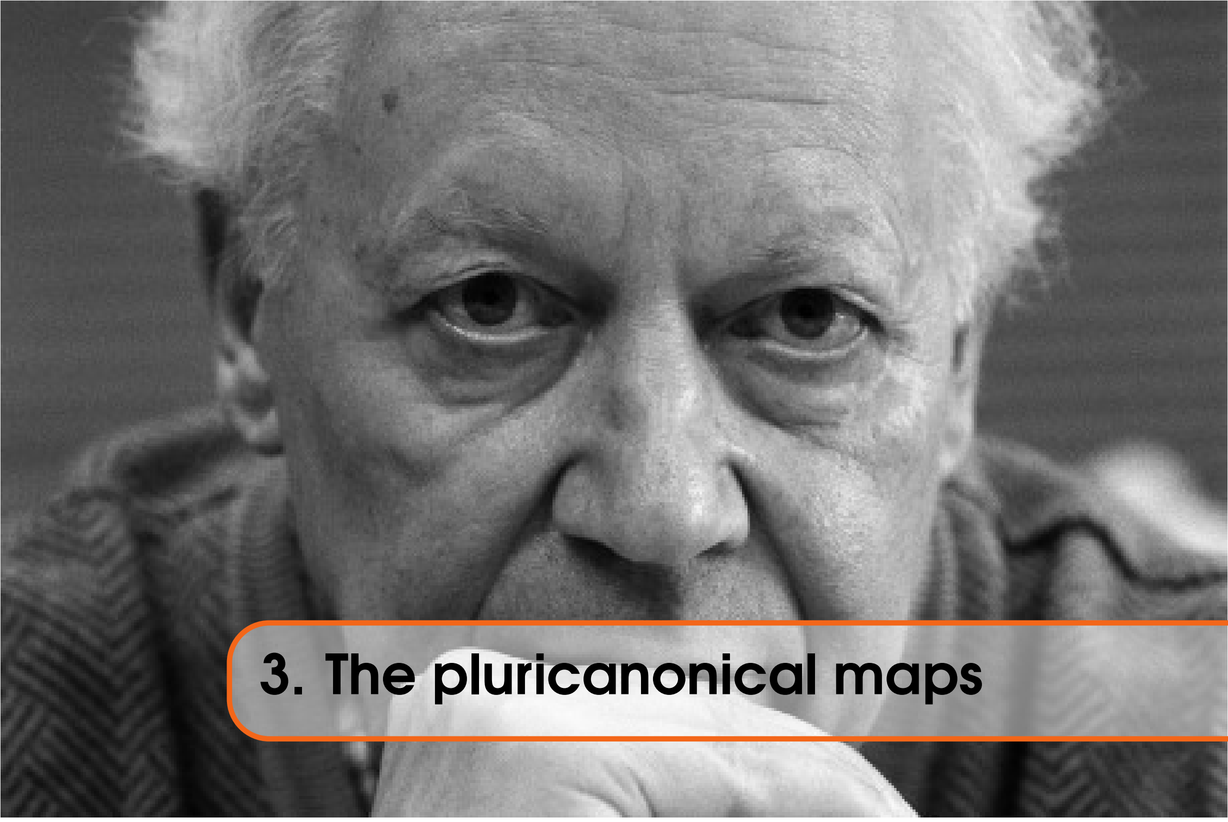
$$(x_1, x_2, x_3, y_1, y_3) \mapsto (ix_1, -x_2, -ix_3, iy_1, -iy_3)$$

where i is a square root of -1. Show that one can choose G_1 $\mathbb{Z}/4\mathbb{Z}$-invariant and G_2 $\mathbb{Z}/4\mathbb{Z}$-antiinvariant (this means that the generator maps the polynomial, say g_2, in $-g_2$), so that X_{16} is smooth and the action is étale. Then show that the quotient surface $X_{16}/\mathbb{Z}/4\mathbb{Z}$ is a minimal surface of general type. Compute its invariants p_g. q and K^2_S and locate it in the geography.

If your computations are correct, you should find the same invariants of another example in these notes. Prove[b] that these surfaces are not isomorphic to those. ■

[a] In case you don't know, the singular locus of $\mathbb{P}$ is the line $(0,0,0,y_1,y_3)$

[b] Hint: Compute fundamental groups



3. The pluricanonical maps

3.1 Is the m-canonical map an embedding?

If S is a minimal surface of general type, as P_m grows very quickly, it is natural to ask if the m-canonical maps $\varphi_{|mK_S|}: S \dashrightarrow \mathbb{P}^{P_m-1}$ are, for large m, embeddings. Note that in the example 2.2, this is true for $m \geq 5$, but fails for smaller m: the $4-$canonical map has degree 2.

On the other hand, if there is a curve C in $\langle K_S \rangle^{\perp}$, there is no hope that one of these maps be an embedding: by Proposition 2.1.3 C is smooth and rational and then $\forall m$, $\mathscr{O}_S(mK_S) \otimes \mathscr{O}_C \cong \mathscr{O}_C$ and then $\varphi_{|mK_S|}$ contracts C to a point.

A classical result claims

Theorem 3.1.1 Let $\{E_1, \dots, E_r\}$ be irreducible curves in a smooth surface S such that the intersection matrix $(E_i \cdot E_j)$ is negative definite. Then there exists a normal surface X and a map $\pi: S \to X$ contracting each E_i to a point p_i so that $p_i = p_j$ if and only if E_i and E_j belong to the same connected component of $\cup E_i$, and mapping biregularly the complement of $\cup E_i$ onto the complement of $\{p_i\}$.

By Proposition 2.1.3 and the Hodge Index Theorem 1.1.1, the set of curves C with $KC = 0$ has the properties required to apply Theorem 3.1.1, and so the next definitions makes sense.

Definition 3.1.1 Let S be a smooth surface of general type. Its **canonical model** is the surface obtained from its minimal model by contracting all curves C with $K_S C = 0$. Canonical models of surfaces of general type are also called **canonical surfaces**.

By the argument above, $\varphi_{|mK_S|}$ factors through the projection onto the canonical model. To understand these maps a bit more, we need to study the singularities of a canonical surface.

3.2 Normal surfaces

Recall that the singular locus of a normal variety has codimension at least 2, and therefore normal surfaces have only finitely many singular points.

Theorem 3.2.1 Let X be a normal surface. Then there is a smooth surface Y and a birational morphism $\pi\colon Y \to X$ such that the preimage of every singular point p of X is a connected divisor.

Definition 3.2.1 Y and the pair (Y,π) are a **resolution of the singularities** of X. We will say that an irreducible and reduced curve $E \subset Y$ is **exceptional** if π maps E to a point.

The resolution is **minimal** if y does not contain any smooth rational curve E with $E^2 = -1$ contracted by π to a point.

It is easy to prove, arguing as in proof of Proposition 1.2.5, that minimal resolutions of the singularities always exists[1].

Definition 3.2.2 A singular point p of a normal surface X is a **Du Val** singularity if there is a resolution of the singularities $\pi\colon S \to X$ so that for each curve $C \subset \pi^{-1}(p)$, C is smooth, rational, and $K_X C = 0$.

So all singular points of a canonical surface are Du Val, that gives us the motivation to classify them.

Definition 3.2.3 A **snc** (=smooth normal crossing) divisor in a surface S is a divisor $C = \sum C_i$ such that the C_i are pairwise distinct smooth irreducible divisor and $\forall i \neq j$ $C_i C_j \leq 1$ (in other words: C_i and C_j are either disjoint or they intersect transversally in a point).

To each snc divisor we associate a graph by picking a vertex v_i for each curve C_i and drawing an edge among the v_i and v_j if and only if $C_i C_j = 1$

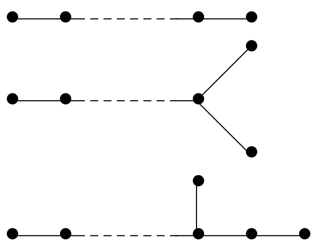
One usually decorates the graph by attributing some numbers to each vertex, namely the genus of the curve and/or its selfintersection. This is useless in our case as we are only interested in snc divisors whose irreducible components are rational with selfintersection -2.

■ **Example 3.1** Here are few examples of graphs which are trees (this means: connected not containing any cycle), which plays an important role in the following. In all cases the subscript n is the number of vertices.

A_n, $n \geq 1$

D_n, $n \geq 4$

E_n, $n \geq 6$



■

Proposition 3.2.2 Let p be a Du Val singularity of a normal surface X, $S \to X$ a minimal resolution of the singularities. Then the preimage of p, taken with the reduced structure, is a smooth normal crossing divisor of type[2] A_n, D_n, E_6, E_7 or E_8.

Proof. We are going to repeatedly use Proposition 2.1.3, and namely that $(C_i \cdot C_j)$ is negative definite.

We consider then the divisor $C = \sum C_i$ sum of the curves contracted to p with multiplicity 1. We know that they are all smooth and rational with $K_S C = 0$. Moreover, if there are two of them with $C_i C_j \geq 2$, then $(C_i + C_j)^2 \geq 0$ contradicts the negative definiteness.

[1]With some more effort one can also prove that the minimal resolution is also unique up to isomorphism. Warning: minimal resolutions of singularities can be defined and exist also in higher dimension, but then uniqueness fails.

[2]That's why these singularities are also known as A-D-E singularities.

So C is an snc divisor, and we can consider its dual graph. At this point we only know that it is connected.

Let $|V|$ be the number of vertices and $|E|$ the number of edges of the graph; then $C^2 = 2(|E|-|V|)$, so Proposition 2.1.3 gives $|E|<|V|$. This property characterizes, among the connected graphs, the trees (connected graphs without cycles). So the graph is a tree.

Recall that the degree of a vertex is the number of edges through it, so the number of curves intersecting it. Consider then the divisor

$$C_i' = 2C_i + \sum_{j|C_iC_j=1} C_j.$$

Then $(C_i')^2 = 2(n-4)$, so Proposition 2.1.3 gives $n \leq 3$.

We say that a vertex of the graph v is a *fork* if $\deg v = 3$. We show that the graph as at most one fork by assuming by contradiction that it has two forks. Then we consider a minimal tree containing the two forks and the three vertices attached to each fork. Consider the divisor $C = \sum c_i C_i$ with $c_i = 0$ if the corresponding vertex is not in this subtree, $c_i = 1$ if it is a leaf of the subgraph, $c_i = 2$ else. Then $C^2 = 0$ contradicting Proposition 2.1.3.

So the graph is a tree with at most one fork. The trees without forks are exactly the graphs A_n. We have then only to consider now trees with exactly one fork, say the vertex v_0. They are union of three branches G_1, G_2 and G_3, that are subgraphs G_i isomorphic to a graph A_{n_i} with v_0 as one leaf, $n_i \geq 2$.

Then we pick the divisor with rational coefficients $C = \sum c_i C_i$ where c_i is, if the vertex of C_i belongs to the branch G_j, $(n_j - d_i)/n_j$ where d_i is the distance of the vertex from v_0. Note that the coefficient of the curve corresponding to the fork is 1. Then a direct computation shows that $C^2 < 0$ is equivalent to

$$\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} > 1$$

whose integral solutions (n_1, n_2, n_3) with $2 \leq n_1 \leq n_2 \leq n_3$ are $(2,2,n)$ for $n \geq 2$ (that's D_{n+2}) and $(2,3,n)$ for $3 \leq n \leq 5$ (that's E_{n+3}). ■

With a bit more effort one can prove [KM98, Theorem 4.22]

Theorem 3.2.3 Let X be a normal surface and $p \in X$ a Du Val singularity.

Then the Zariski tangent space of X has dimension 3, and a p is locally analytically determined by the dual graph of the exceptional divisor of the minimally resolution of its singularity.

More precisely an analytic neighbourhood of p is biholomorphic to a neighbourhood of the origin of one of the following hypersurfaces of $\mathbb{C}^3$:

$$x^2 + y^2 + z^{n+1} = 0 \text{ if the graph is } A_n;$$

$$x^2 + y^2 z + z^{n-1} = 0 \text{ if the graph is } D_n;$$

$$x^2 + y^3 + z^4 = 0 \text{ if the graph is } E_6;$$

$$x^2 + y^3 + yz^3 = 0 \text{ if the graph is } E_7;$$

$$x^2 + y^3 + z^5 = 0 \text{ if the graph is } E_8.$$

Definition 3.2.4 Let X be a normal surface. Then we may remove the singular points, and consider the smooth part X° of X: the zero locus of a $2-$form on it is a canonical divisor K_{X° of X°. Its Zariski closure is a Weil divisor on X which we will denote by K_X, a **canonical divisor** of X.

R Warning: K_X may be not Cartier.

Proposition 3.2.4 Let X be a canonical surface. Then K_X is Cartier. If $S \to X$ is the map from the minimal model, solving the singularities of X, then $\pi^* K_X = K_S$. Moreover $h^i(mK_X) = 0$, $\forall m \neq \{0,1\}$.

Proof. K_X is Cartier since all singular points have embedded dimension 3 by Theorem 3.2.3 . By the definition of K_X, $K_S = \pi^* K_X + E$ for some $E = \sum e_i E_i$ when E_i are exceptional and so $(E_i \cdot E_j)$ is (negative) definite. From $K_S E_i = 0$, then $EE_i = 0$ which immediately implies that $\forall i$, $e_i = 0$, so $E = 0$.

The vanishing is proved as in Corollary 2.1.2 by Mumford's vanishing theorem (on normal surfaces). ■

3.3 Bombieri's theorem on the 5-canonical map

We will need the following, a simplified version of ([Cat+99, Theorem 1.1]).

Theorem 3.3.1 — Curve embedding theorem. Let C be an effective Weil divisor in a normal surface X, H a Cartier divisor on C. If for every subcurve $B \subset C$

$$HB \geq 2p_a(B) + 1$$

then H is very ample[a].

[a] H is defined only on C, so the claim is that $H^0(\mathscr{O}_C(H))$ embeds C. Indeed X does not play any role in the statement, and the theorems holds more generally for C a scheme of pure dimension 1 with certain properties, and effective Weil divisors in normal surfaces are just a special case. The intersection number HB is defined as the degree of the line bundle $\mathscr{O}_C(H) \otimes \mathscr{O}_B$: If H is the restriction of a Cartier divisor H' on X, then $HB = H'B$.

R If C is smooth of genus g, then the assumption becomes $\deg H \geq 2g+1$ and the statement follows by Riemann-Roch.

Indeed H is very ample if and only if for every **cluster**[3] of length two contained in C the restriction map

$$H^0(\mathscr{O}_C(H)) \to H^0(\mathscr{O}_C(H) \otimes \mathscr{O}_Z) \cong \mathbb{C}^2$$

is surjective (*i.e.*: *the map induced by H separates each pair of points*).

If C is a smooth curve then the statement follows immediately since, by Serre duality (writing Z as a divisor on C), both H and $H-Z$ are not special (having degree $\geq 2g-1$), and therefore by Riemann-Roch and Serre duality $h^0(\mathscr{O}_C(H)) - h^0(\mathscr{O}_C(H-Z)) = \chi(\mathscr{O}_C(H)) - \chi(\mathscr{O}_C(H-Z)) = \deg Z = 2$.

This is a simplified version of a theorem proved by Bombieri in [Bom73]. We give here the proof of [Cat+99].

Theorem 3.3.2 — Bombieri's theorem on the 5−canonical map. Let X be a canonical surface. Then if $m \geq 5$ then mK_X is very ample.

Proof. The claim is that mK_X is very ample, that is that for every cluster $Z \subset X$ of degree 2 the evaluation map $H^0(\mathscr{O}_X(mK_X)) \to H^0(\mathscr{O}_Z(mK_X)) \cong \mathbb{C}^2$ is surjective. Each curve C in X containing Z allows us to split that map as a composition

$$H^0(\mathscr{O}_X(mK_X)) \to H^0(\mathscr{O}_C(mK_X)) \to H^0(\mathscr{O}_Z(mK_X))$$

[3] a cluster is a scheme of pure dimension zero

and we will find a curve C such that both the above maps are surjective: then their composition will be surjective too, proving the claim.

First we construct C, by picking a curve in $|(m-2)K_X|$ containing Z. Indeed, by Riemann-Roch theorem, since by Corollary 2.1.2 $\forall i>0$, $\forall m\geq 2$, $h^i(\mathscr{O}_S(mK_S))=0$,

$$h^0(\mathscr{O}_X((m-2)K_X))=h^0(\mathscr{O}_S((m-2)K_S))=\chi(\mathscr{O}_S((m-2)K_S))=\chi(\mathscr{O}_S)+\binom{m-2}{2}K_S^2\geq 1+3=4$$

and then $h^0(\mathscr{I}_Z\mathscr{O}_X((m-2)K_X))\geq 4-2>0$: such a C exists.

Then we need the surjectivity of the map $H^0(\mathscr{O}_X(mK_X))\to H^0(\mathscr{O}_C(mK_X))$: this is obvious by the long cohomology exact sequence associated to the short exact sequence

$$0\to\mathscr{O}_X(mK_X-C=2K_X)\to\mathscr{O}_X(mK_X)\to\mathscr{O}_C(mK_X)\to 0$$

since $h^1(\mathscr{O}_X(2K_X))=0$ by Proposition 3.2.4.

Finally we prove the surjectivity of the evaluation map $H^0(\mathscr{O}_C(mK_X))\to H^0(\mathscr{O}_Z(mK_X))$ by the curve embedding theorem. Indeed, if $\mathscr{O}_C(mK_X)$ is very ample, then clearly the evaluation map on Z (or any other cluster of degree 2 in C) is surjective. We only then need to prove that for every subcurve B of C

$$mK_XB\geq 2p_a(B)+1$$

If $B=C$ then

$$mK_XC=(K_X+K_X+C)C=K_XC+2p_a(C)-2=(m-2)K_X^2+2p_a(C)-2\geq 3+2p_a(C)-2.$$

We can then assume that B is a proper subcurve of C. Note that, if $\tilde{B}$ is a lift of B to X, then $K_XB=K_S\tilde{B}\geq 1$, since π contracts all curves in $\langle K_S\rangle^{\perp}$ and then it is enough to show

$$(K_X+C)B\geq 2p_a(B) \tag{3.1}$$

To prove (3.1) let us assume, for sake of simplicity, X smooth. Then, writing $C=A+B$, as $2p_a(B)=(K_X+B)B+2$, the statement to prove is just $AB\geq 2$. In other words, we have to prove that C is *2-connected*.

We assume then, by contradiction, $AB\leq 1$. Note that $C^2>0$, and therefore, by the Hodge Index Theorem 1.1.1

$$A^2B^2\leq (AB)^2 \tag{3.2}$$

with equality possible if and only if A and B are numerically proportional. As X is a canonical surfaces (no curves in $\langle K_S\rangle^{\perp}$), $0<(m-2)K_SA=CA=A^2+AB$, so $A^2>-AB$, and similarly $B^2>-AB$.

If $AB\leq 0$ this contradicts (3.2). Then $AB=1$, by (3.2) $\min(A^2,B^2)\leq 1$. If $A^2\leq 1$

$$1\leq K_XA=\frac{CA}{m-2}=\frac{A^2+AB}{m-2}\leq\frac{2}{m-2}\leq\frac{2}{3}$$

a contradiction. If $B^2\leq 1$ we get a similar contradiction by considering K_XB.

We have concluded the proof under the assumption that the canonical model X be smooth. The general case can be proved in a similar way by considering the minimal model S and by lifting C and B to S. We skip the details, only mentioning that one has to carefully choose the lifting of B. ■

R Theorem 3.3.2 is a major tool for the proof of the existence of a quasi-projective coarse moduli space of canonical surfaces with given invariants K^2,p_g,q. Indeed using the 5-canonical embeddings one find all these surfaces in a family parametrized by a suitable Hilbert scheme.

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